Linear transformations

If A is an $m \times n$ matrix, we defined the matrix transformation induced by A $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by

$$T_A(\vec{x}) = A\vec{x}.$$

Today we will give a different definition of transformations and then show that the two definitions are actually equivalent.

Definition A function $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation if it satisfies the following properties for all vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ and all scalars a:

① $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

② $T(a\vec{x}) = aT(\vec{x})$.

Note: This definition implies that $T(0) = T(0\vec{x}) = 0T(\vec{x}) = 0$ so $T(0)=0$ for all linear transformations.

We can easily check that transformations induced by matrices satisfy these properties:

If A is a matrix, then

$$T_A(\vec{x}+\vec{y}) = A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = T_A(\vec{x}) + T_A(\vec{y}).$$

$$T_A(a\vec{x}) = A(a\vec{x}) = aA\vec{x} = aT_A(\vec{x}).$$

Ex: Define $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 \end{bmatrix}.$$

Is this a linear transformation?

① $$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = T\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix} = \begin{bmatrix} (x_1+y_1) + (x_2+y_2) \\ (x_3+y_3) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ y_3 \end{bmatrix} = T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + T\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}. \quad \checkmark$$

② $$T\left(a\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T\begin{bmatrix} a x_1 \\ a x_2 \\ a x_3 \end{bmatrix} = \begin{bmatrix} a x_1 + a x_2 \\ a x_3 \end{bmatrix} = a\begin{bmatrix} x_1 + x_2 \\ x_3 \end{bmatrix}$$

$$= a\,T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \quad \checkmark$$

So, yes. It's a linear transformation.

We can also see that T satisfies our original definition of transformation by finding a matrix $\underset{\substack{\uparrow \\ 2\times 3}}{A}$ such that $T(\vec{x}) = A\vec{x}$.

$$\underset{\substack{\shortparallel \\ A}}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 \end{bmatrix}$$

So $$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\vec{x}.$$

Theorem: A function $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation if and only if there is a matrix A such that $T = T_A$. (i.e. $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$.)

In this case the matrix A is unique and is given by

$$A = \left[T(\vec{e}_1) \; T(\vec{e}_2) \; \dots \; T(\vec{e}_n) \right]$$

i.e. its i^{th} column is $T(\vec{e}_i)$, where $\vec{e}_i$ = i^{th} column of identity matrix (1 in i^{th} entry, 0 elsewhere).

($\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ is called the standard basis of $\mathbb{R}^n$)

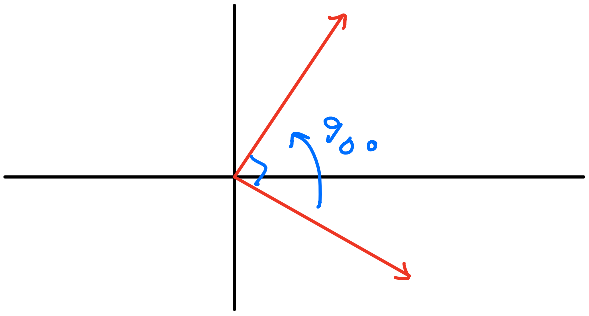
Ex: Going back to example above,

$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 \end{bmatrix}$, we can use the above formula to construct the matrix.

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underset{\substack{\uparrow \\ \text{column 1}}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \quad T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \underset{\substack{\uparrow \\ \text{column 2}}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \quad T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underset{\substack{\uparrow \\ \text{column 3}}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}},$$

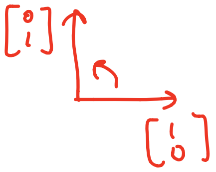
so the associated matrix is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

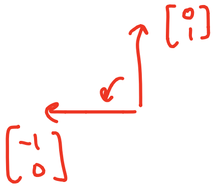
Ex: Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the transformation given by counterclockwise rotation by 90°.



We already showed this is a linear transformation, but now we have an easy way to compute the corresponding matrix:

$$T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix} \quad \text{and} \quad T\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-1\\0\end{bmatrix}$$





so $T(\vec{x})=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\vec{x}$.

Ex: Is $T:\mathbb{R}^2\longrightarrow\mathbb{R}^2$, defined

$$T\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1-x_2\\1\end{bmatrix}$$

a linear transformation?

No $\quad T\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\neq\begin{bmatrix}0\\0\end{bmatrix}$

We can also see this since

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}y_1\\y_2\end{bmatrix}\right)=T\begin{bmatrix}x_1+y_1\\x_2+y_2\end{bmatrix}=\begin{bmatrix}(x_1+y_1)-(x_2-y_2)\\1\end{bmatrix}$$

$$\neq$$

$$T\begin{bmatrix}x_1\\x_2\end{bmatrix}+T\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}(x_1-x_2)+(y_1-y_2)\\2\end{bmatrix}$$

Linear combinations

One important property of linear transformations is that they preserve linear combinations.

That is, if $\vec{x}_1,\dots,\vec{x}_k$ are vectors, $a_1,\dots,a_k$ scalars, and $T:\mathbb{R}^n\to\mathbb{R}^m$ a linear transformation, then

$$T\left(a_1\vec{x}_1 + a_2\vec{x}_2 + \dots + a_k\vec{x}_k\right)$$

$$= T\left(a_1\vec{x}_1\right) + T\left(a_2\vec{x}_2\right) + \dots + T\left(a_k\vec{x}_k\right)$$

$$= a_1 T\left(\vec{x}_1\right) + a_2 T\left(\vec{x}_2\right) + \dots + a_k T(\vec{x}_k).$$

This means that if we know $T(\vec{x}_1), \dots, T(\vec{x}_k)$, then we can find $T(\vec{y})$ for any linear combination $\vec{y}$ of $\vec{x}_1, \dots, \vec{x}_k$.

Ex: Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and

$$T\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix}, \quad T\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}.$$

What is $T\begin{bmatrix}3\\3\end{bmatrix}$?

We first want to write $\begin{bmatrix}3\\3\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}-1\\2\end{bmatrix}$:

$$a\begin{bmatrix}1\\-1\end{bmatrix} + b\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix} \Rightarrow \begin{matrix} a - b = 3 \\ -a + 2b = 3 \end{matrix} \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 3 \\ -1 & 2 & 3 \end{array}\right]$$

$$\xrightarrow{② + ①} \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & 6 \end{array}\right] \xrightarrow{① + ②} \left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 6 \end{array}\right] \Rightarrow a = 9,\ b = 6$$

So $\begin{bmatrix}3\\3\end{bmatrix} = 9\begin{bmatrix}1\\-1\end{bmatrix} + 6\begin{bmatrix}-1\\2\end{bmatrix}$, so we get

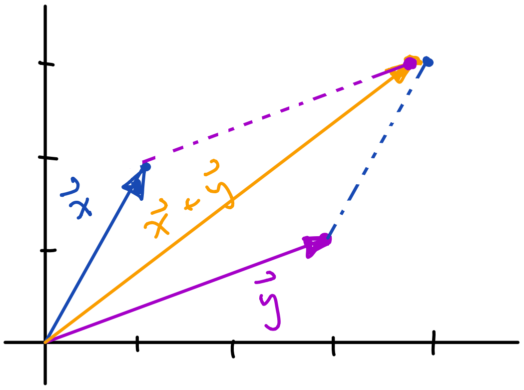
$$T\begin{bmatrix}3\\3\end{bmatrix} = T\left(9\begin{bmatrix}1\\-1\end{bmatrix} + 6\begin{bmatrix}-1\\2\end{bmatrix}\right) = 9\,T\begin{bmatrix}1\\-1\end{bmatrix} + 6\,T\begin{bmatrix}-1\\2\end{bmatrix}$$

$$= 9\begin{bmatrix}5\\1\end{bmatrix} + 6\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}45\\9\end{bmatrix} + \begin{bmatrix}0\\6\end{bmatrix} = \begin{bmatrix}45\\15\end{bmatrix}.$$

Geometry of linear transformations

If $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^2$, we can describe their sum geometrically.

Ex: $\vec{x} = \begin{bmatrix}1\\2\end{bmatrix}$, $\vec{y} = \begin{bmatrix}3\\1\end{bmatrix}$, $\vec{x}+\vec{y} = \begin{bmatrix}4\\3\end{bmatrix}$



Notice that $\vec{x}+\vec{y}$ is the diagonal of the parallelogram formed by taking $\vec{x}$ and $\vec{y}$ as two of the sides.

Equivalently, $\vec{x}+\vec{y}$ is the vector created by putting $\vec{x}$ and $\vec{y}$ end-to-end.

Rotations

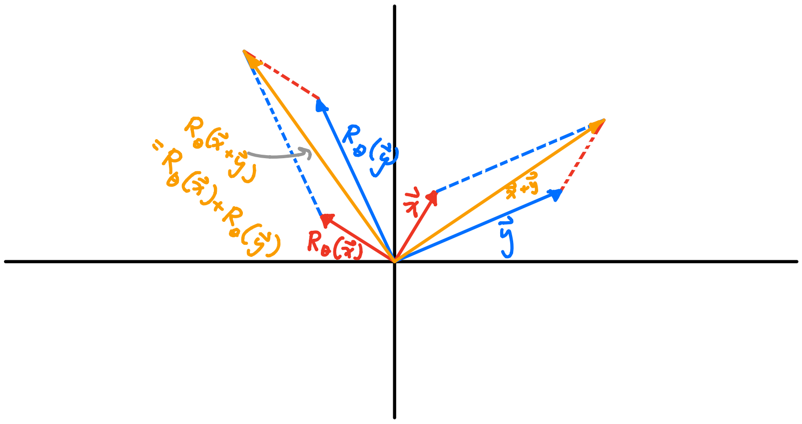
Let θ be an angle (in radians) and define

$$R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

to be counterclockwise rotation in the plane.

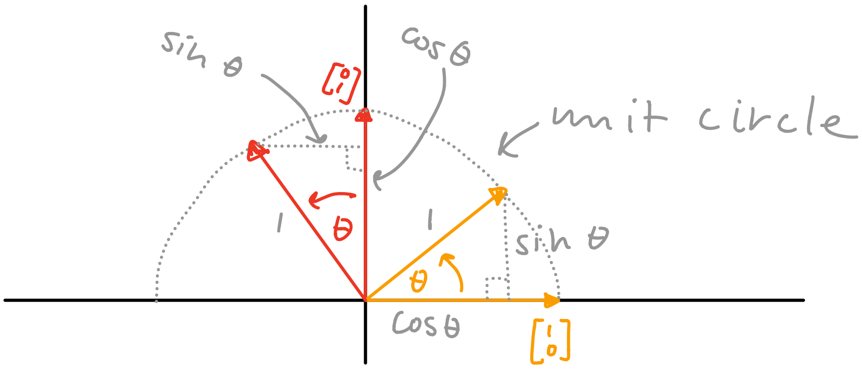
We can check that R_θ is actually a linear transformation:

If $\vec{x}, \vec{y}$ are in $\mathbb{R}^2$, then $\vec{x}+\vec{y}$ is the diagonal of the parallelogram formed by $\vec{x}$ and $\vec{y}$. Thus, if we rotate the parallelogram by θ, the diagonal of the parallelogram formed by $R_\theta(\vec{x})$ and $R_\theta(\vec{y})$ will be $R_\theta(\vec{x}+\vec{y})$, but also $R_\theta(\vec{x})+R_\theta(\vec{y})$.
Thus, $R_\theta(\vec{x}+\vec{y}) = R_\theta(\vec{x})+R_\theta(\vec{y})$.



If a is a scalar, then $a\vec{x}$ is a vector a times as long as $\vec{x}$ in the same direction as $\vec{x}$, so $R_\theta(a\vec{x})$ will be a times as long as $R_\theta(\vec{x})$, in the same direction as $R_\theta(\vec{x})$, so $R(a\vec{x}) = aR(\vec{x})$.

Thus, R_θ is linear, and we can calculate the corresponding matrix by plugging in the standard basis:

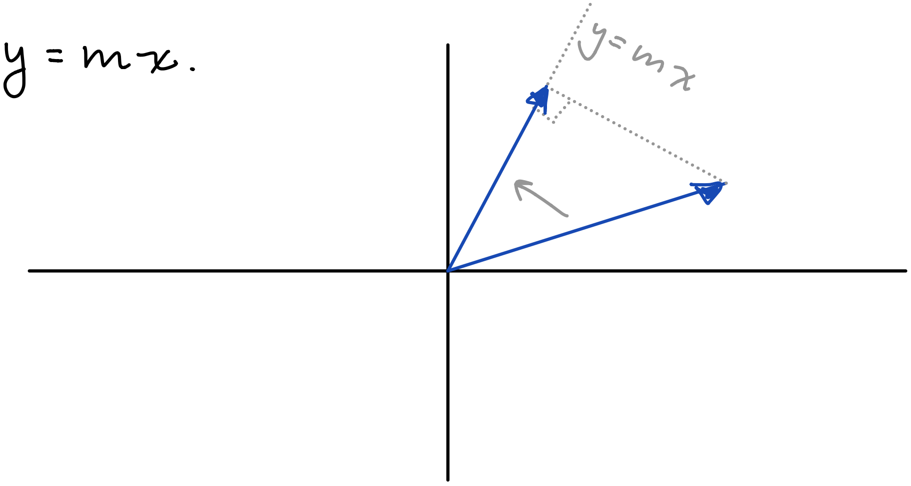


so $R_\theta \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ and $R_\theta \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$.

Thus, R_θ is the linear transformation induced by the matrix $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$.

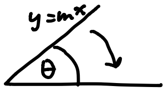
Projection

Let $P_m : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ denote projection onto the line $y = mx$.

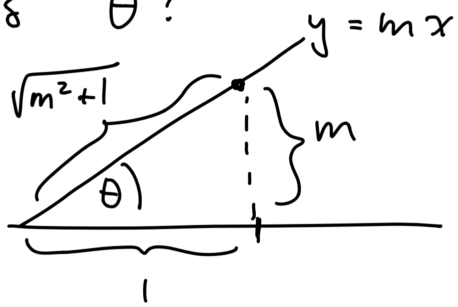


You can check that this is indeed a linear transf.
How to find its matrix?

Trick: ① Rotate so that $y=mx$ is the x-axis (what is θ?)
② Project onto x-axis
③ Rotate back (by θ).



First, what is θ?



$$\cos\theta = \frac{1}{\sqrt{m^2+1}}$$

$$\sin\theta = \frac{m}{\sqrt{m^2+1}}$$

Projecting onto x-axis is just taking the x-coordinate, so the corresponding matrix is $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$.

Thus, $P_m = \underset{\substack{\uparrow\\ \text{rotate}\\ \text{ccw by } \theta}}{R_\theta} \circ \underset{\substack{\uparrow\\ \text{project onto}\\ x\text{-axis}}}{P_0} \circ \underset{\substack{\nwarrow\\ \text{rotate}\\ \text{clockwise}\\ \text{by } \theta}}{R_{-\theta}}$

So to get the corresponding matrix, we multiply the 3 corresponding matrices:

$$\underbrace{\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}}_{\begin{gathered}\shortparallel\\ \begin{bmatrix}\cos\theta & 0\\ \sin\theta & 0\end{bmatrix}\end{gathered}} \underbrace{\begin{bmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{bmatrix}}_{\begin{gathered}\shortparallel\\ \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\end{gathered}}$$

$$= \begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix} = \begin{bmatrix}\frac{1}{m^2+1} & \frac{m}{m^2+1}\\ \frac{m}{m^2+1} & \frac{m^2}{m^2+1}\end{bmatrix}$$

Let $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line $y=mx$. similar analysis shows that the corresponding matrix is $\frac{1}{1+m^2}\begin{bmatrix}1-m^2 & 2m\\ 2m & m^2-1\end{bmatrix}$

Practice problems: 2.6 : 2, 3, 7, 9, 12abc